6. Generating functions. Fibonacci numbers and linear recurrence relations

6.1. **Examples of generating functions.** Consider the following two examples.
Example 1. Consider the sequence $a_n = n+1$, $n \in \mathbb{Z}_{\geq 0}$. Then the generating function is

$$A(x) = 1 + 2x + 3x^2 + \ldots = \frac{d}{dx}(1 + x + x^2 + \ldots) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2}.$$

Example 2. Consider the sequence $b_n = (n+1)^2$, $n \in \mathbb{Z}_{\geq 0}$. Arguing in a similar way, one gets that the generating function is $B(x) = \frac{d}{dx}A(x) - A(x)$.

6.2. **Fibonacci sequence.** The Fibonacci sequence $(F_n)_{n\geq 0}$ is defined by the following recursive formula:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \forall n \geq 2.$$

Another way to interpret the Fibonacci sequence is the following: let S_n denote the number of ways in which one can climb n stairs if allowed to jump one or two stairs at a time. This is the same as to count the number of the solutions of the equation $x_1 + \ldots + x_k = n$ where $x_i \in \{1, 2\}$ and the number k is not fixed. We observe that $S_1 = 1$, $S_2 = 2$ and $S_{n+2} = S_{n+1} + S_n$ for all $n \in \mathbb{Z}_{\geq 1}$. Therefore, we have $S_n = F_{n+1}$.
Identities for Fibonacci numbers. The sum of the first n numbers of the Fibonacci sequence, is

$$\sum_{k=0}^{n} F_k = F_{n+2} - 1.$$

Exercise 6. Prove the following identities for Fibonacci numbers:

$$\begin{aligned}
&(a) \quad F_1 + F_3 + F_5 \ldots + F_{2n-1} = F_{2n} \\
&(b) \quad F_{2n+1} = 3F_{2n-1} - F_{2n-3} \\
&(c)^* \quad F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b.
\end{aligned}$$

Explicit formula for Fibonacci numbers. We want to find an explicit formula for the value of the n-th Fibonacci number. We will present several possible ways to do that.
Method 1.
We will use the generating functions. Let $F(x)$ denote the generating function of the Fibonacci sequence $(F_0, F_1, \ldots)$ that is

$$F(x) = F_0 + F_1x + F_2x^2 + F_3x^3 + \ldots.$$

Note that the convergence radius of this series is at least $\frac{1}{2}$. Multiplying $F(x)$ by x, respectively x^2, we obtain that

$$\begin{aligned}
xF(x) &= F_0x + F_1x^2 + F_2x^3 + F_3x^4 + \ldots \\
x^2F(x) &= F_0x^2 + F_1x^3 + F_2x^4 + F_3x^5 + \ldots.
\end{aligned}$$

Recall that for every $n \geq 2$, we have $F_n = F_{n-1} + F_{n-2}$ and consider $F(x) - xF(x) - x^2F(x)$. Grouping together the coefficients of x^k for every k, one obtains that

$$F(x) - xF(x) - x^2F(x) =$$

$$= F_0 + x(F_1 - F_0) + x^2(F_2 - F_1 - F_0) + x^3(F_3 - F_2 - F_1) + \ldots + x^k(F_k - F_{k-1} - F_{k-2}) + \ldots.$$

This implies $F(x) - xF(x) - x^2F(x) = x$ and thus

$$F(x) = \frac{x}{1 - x - x^2}$$

This means, the general term is

$$F_n = \frac{F^{(n)}(0)}{n!}$$

where $F^{(n)}(0)$ is the value in 0 of the n-th derivative of $F(x)$. We factor $1 - x - x^2$ as $-(x - x_1)(x - x_2)$, where $x_{1,2} = \frac{-1\pm\sqrt{5}}{2}$ This means

$$F(x) = \frac{x}{1 - x - x^2} = \frac{A}{x - x_1} + \frac{B}{x - x_2} = \frac{A(x - x_2) + B(x - x_1)}{-(1 - x - x^2)}$$

From this we obtain that

$$A + B = -1 \quad \text{and} \quad Ax_2 + Bx_1 = 0.$$

This is a system of two equations with A and B as unknowns, so we can obtain exact values for A and B:

$$A = \frac{x_1}{\sqrt{5}} \qquad B = \frac{-x_2}{\sqrt{5}}.$$

One can obtain that:

$$\begin{aligned}
F(x) =& \frac{A}{x - x_1} + \frac{B}{x - x_2} = -\frac{A}{x_1}\frac{1}{1 - \frac{x}{x_1}} - \frac{B}{x_2}\frac{1}{1 - \frac{x}{x_2}} = \\
=& -\frac{A}{x_1}\sum_{n=0}^{\infty} x_1^{-n}x^n - \frac{B}{x_2}\sum_{n=0}^{\infty} x_2^{-n}x^n \\
=& \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty} x_1^{-n}x^n - \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty} x_2^{-n}x^n. \\
=& \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\left(x_1^{-n} - x_2^{-n}\right)x^n.
\end{aligned}$$

This implies that the general term F_n is

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

Method 2.
We look first for a geometric series that satisfies $F_n = F_{n-1} + F_{n-2}$, that is $F_n = c \cdot \alpha^n$ for all $n \in \mathbb{Z}_{\geq 0}$. This implies that $c\alpha^n = c\alpha^{n-1} + c\alpha^{n-2}$ and thus $\alpha^2 - \alpha - 1 = 0$. Solving this quadratic equation, we get $\alpha_{1,2} = \frac{1\pm\sqrt{5}}{2}$. Next, we search for F_n in the form

$$F_n = c_1\alpha_1^n + c_2\alpha_2^n = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for some $c_1, c_2 \in \mathbb{R}$. The initial conditions imply

$$\begin{aligned}
F_0 =& c_1 + c_2 = 0 \\
F_1 =& c_1\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right) = 1.
\end{aligned}$$

Thus, the only solution is

$$c_1 = \frac{1}{5} \quad c_2 = \frac{-1}{5}.$$

Hence we find

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

6.3. **Linear recurrence relations.** In general, to solve linear recurrence relations of the form

$$a_{n+k} = c_{k-1}a_{n+k-1} + \ldots + c_0 a_n$$

we have the following recipe. Denote by $\lambda_1, \ldots \lambda_s$ the (possibly complex) roots of the equation

$$\lambda^k = c_{k-1}\lambda^{k-1} + \ldots + c_0$$

where λ_i has multiplicity k_i and $\sum_{i=1}^{s} k_i = k$.

Theorem 6.1. *A formula for a_n is the solutions to the recurrence above if and only if it has the form $a_n = \sum_{i=1}^{s} P_i(n)\lambda_i^n$, where each $P_i(n)$ is a polynomial of degree $k_i - 1$ with coefficients chosen arbitrarily. Moreover, for any set of initial values $a_0, \ldots, a_{k-1}$ one can find coefficients of the polynomials $P_i(n)$ so that the solution fits to the initial values. Note that the number of coefficients to be determined is equal to k, the number of initial values.*